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On the SHASTA FCT Algorithm for the Equation $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(v(\rho)\rho) = 0$

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Abstract. In recent years, Boris, Book and Hain have proposed a family of finite difference methods called FCT techniques for the Cauchy problem of the continuity equation. The purpose of this paper is to study the stability and convergence about the SHASTA FCT algorithm, which is one of the basic schemes among many FCT techniques, though not in its original form but a slightly modified one for our technical reason. (Our numerical experiments indicate less distinction between the algorithm dealt with here and the original SHASTA FCT one in terms of reproduction of sharp discontinuities.) The main results are Theorems 1 and 2 concerning the L^∞-stability and the L^1_{loc}-convergence, respectively.

1. Introduction. There have been proposed many finite difference schemes for the initial-value problem of the conservation law:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} g(\rho) = 0. \tag{1}$$

([5], [7], [8], [11], [12], [13], [14], [18], [19], [20] among others.)

But high-order methods, Lax-Wendroff's scheme for example, are known to yield in some cases numerical solutions which approximate nonphysically relevant solutions, that is, those which violate the entropy condition. The development of overshoot, undershoot, and excessive oscillation is another problem. On the other hand, the solutions of some "positive" schemes converge to the generalized solution satisfying the entropy condition, but they are necessarily of first-order accuracy.

In recent years, Boris, Book, and Hain ([1], [2], [3], and [4]) have proposed a family of finite difference methods called "the flux-corrected transport (FCT)" techniques for the hyperbolic equation, which is one of the special cases of (1),

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(v(\rho) \cdot \rho) = 0 \qquad (|x| < \infty,\ t > 0) \tag{2}$$

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with the initial condition of

(3) $$\rho(x, 0) = \rho_0(x) \qquad (|x| < \infty).$$

For simplicity, we shall denote problem (2) with (3) by (CP).

The FCT technique consists of a finite difference scheme and a nonlinear antidiffusion operation. Given the numerical solution $\{\rho_j^n\}_{j=0,\pm 1,\dots}$ at time step $t = n\tau$, one calculates the temporary solution $\{\bar{\rho}_j^{n+1}\}_{j=0,\pm 1,\dots}$ by the specified difference scheme, then applies the antidiffusion operation to it to obtain $\{\rho_j^{n+1}\}_{j=0,\pm 1,\dots}$ at time step $t = (n + 1)\tau$. The essential part of the technique is characterized by the latter which removes excessive diffusion contained in the temporary solution by the former, thus reproduces relatively sharp wave shapes. Thanks to this antidiffusion operation, solutions by the FCT technique have a distinguishing property which cannot be expected by a sole use of finite difference schemes.

In their papers, they develop many FCT techniques and compare their methods numerically with the two-step Lax-Wendroff scheme, the leap-frog scheme, and the one-sided scheme in the case of square waves, and in addition they mention some applications to more complicated problems. Further, with the aid of Fourier analysis, they investigate the amplitude and phase errors and the Gibbs phenomenon in the special case where $\rho_0(x)$ is a Fourier harmonic function and $v(\rho)$ is a constant function. For this argument, however, they omit nonlinear characteristics of the antidiffusion operation.

The purpose of this paper is to study the stability and convergence about the SHASTA FCT algorithm, one of the basic schemes among many FCT techniques, for the full nonlinear problem (CP). (The term SHASTA stands for "SHarp And Smooth Transport Algorithm".) For technical purposes, we modify the original SHASTA scheme slightly. And we reconstruct the FCT part so that it may be *monotone*. Numerically, our version keeps the same property in the sharp reproduction of discontinuities as that of the original one. Our assumptions are that $v(\rho)$ is a smooth function of the single real variable ρ and that $\rho_0(x)$ is a bounded function, and we shall further assume that $\rho_0(x)$ is a measurable function having locally bounded variation when we will deal with the convergence of solution. The main results are Theorems 1 and 2, which describe the stability in the L^∞-sense and the convergence of a subsequence to a generalized solution in the L^1_{loc}-sense, respectively.

2. SHASTA FCT **Algorithm**. We review here the SHASTA FCT algorithm, which is slightly modified for our convenience. Conceptually, the FCT technique consists of three operations: a *transport* and a *diffusion* followed by an *antidiffusion*. But in the present case the transport and diffusion are performed as a single finite difference operation SHASTA. Let the half-space $\mathbf{R} \times \mathbf{R}^+ = \{(x, t): -\infty < x < +\infty, t \geq 0\}$ be covered by a grid defined by the straight lines

$$x = jh, \qquad t = k\tau,$$

where h and τ are fixed real numbers, k runs over the nonnegative integers and j

assumes all integral values. We use the following notations:

$$\begin{cases} \rho_j^k = \rho_h(jh, k\tau), \qquad \bar{\rho}_j^k = \bar{\rho}_h(jh, k\tau), \\ \rho_h(x, t) = \rho_j^k \quad \text{for } jh \leqslant x < (j+1)h \text{ and } k\tau \leqslant t < (k+1)\tau. \end{cases}$$

∘ SHASTA. Denote τ/h by λ. We introduce two real-valued functions $E(\xi, \eta)$ and $F(\xi, \eta)$ of two real variables such that

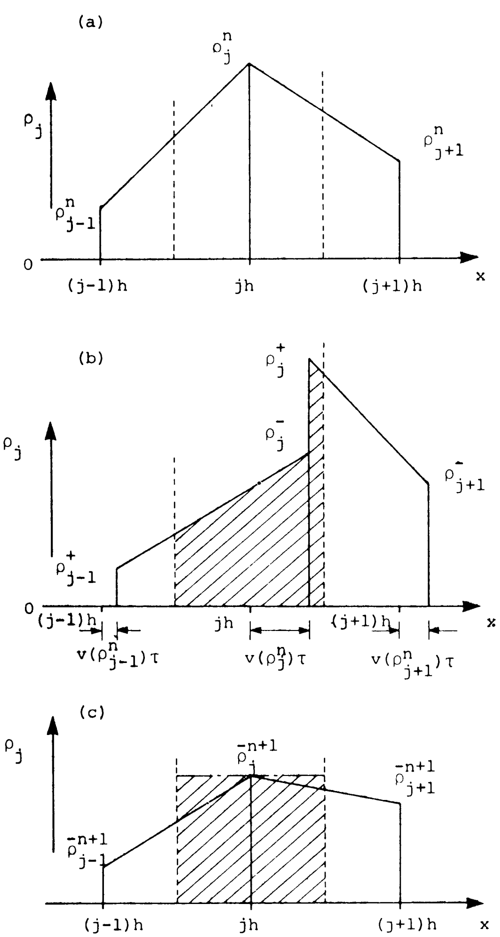
$$E(\xi, \eta) = \lambda(v(\xi) + v(\eta))/(1 + \lambda v(\eta) - \lambda v(\xi)), \tag{4}$$

$$F(\xi, \eta) = \frac{1}{8}\left[(1 + E(\xi, \eta))^2 \xi - (1 - E(\xi, \eta))^2 \eta\right]. \tag{5}$$

The finite difference scheme SHASTA is defined in the form

$$\bar{\rho}_j^{n+1} = \rho_j^n - F(\rho_j^n, \rho_{j+1}^n) + F(\rho_{j-1}^n, \rho_j^n). \tag{6}$$

This formula has a geometrical interpretation as in Figure 1.



Trapezoids representing the shape of fluid elements at $t = n\tau$.

Trapezoids after "transport" operation. The ρ_{j+1}^- and ρ_j^+ are determined so that the relation

$$\begin{cases} (\rho_j^+ + \rho_{j+1}^-)\,(h + v(\rho_{j+1}^n)\tau - v(\rho_j^n)\tau) = (\rho_j^n + \rho_{j+1}^n)h, \\ \rho_j^+ : \rho_j^n = \rho_{j+1}^- : \rho_{j+1}^n \end{cases}$$

holds.

Trapezoids after "diffusion" operation. The $\bar{\rho}_j^{n+1}$ is determined so that the shaded area in (c) is equal to the one in (b).

FIGURE 1
Geometrical interpretation of the SHASTA operation

∘ FCT. The right-hand side of (6) contains a diffusion term $(1/8)(\rho_{j-1}^n - 2\rho_j^n + \rho_{j+1}^n)$ which is velocity-independent. The nonlinear antidiffusion operation FCT to cancel this excessive diffusion is as follows:

$$\rho_j^{n+1} = \bar{\rho}_j^{n+1} - f_{j+1/2}^{n+1} + f_{j-1/2}^{n+1}. \tag{7}$$

Here, $f_{j+1/2}^{n+1}$, which is the antidiffusion flux of our definition, must satisfy the following conditions:

(a) There exists a positive-valued continuous function $K(\nu_1, \nu_2, \nu_3, \nu_4)$ such that $f_{j+1/2}^{n+1} = K(\bar{\rho}_{j-1}^{n+1}, \bar{\rho}_j^{n+1}, \bar{\rho}_{j+1}^{n+1}, \bar{\rho}_{j+2}^{n+1}) \cdot \Delta_{j+1/2}^{n+1}$ in which $\Delta_{j+1/2}^{n+1} \equiv \bar{\rho}_{j+1}^{n+1} - \bar{\rho}_j^{n+1}$.

(b) $\sup_{(\nu_1,\nu_2,\nu_3,\nu_4)\in R^4} [K(\nu_1, \nu_2, \nu_3, \nu_4)] = K_0 \leqslant 1/8$.

(c) $|f_{j+1/2}^{n+1}| \leqslant \mathrm{Min}(|\Delta_{j-1/2}^{n+1}|, |\Delta_{j+3/2}^{n+1}|)$.

(d) If $\Delta_{j-1/2}^{n+1} \cdot \Delta_{j+1/2}^{n+1} \leqslant 0$, then $f_{j-1/2}^{n+1} = f_{j+1/2}^{n+1} = 0$.

(e) If $\bar{\rho}_j^{n+1} \leqslant \bar{\rho}_{j+1}^{n+1}$ (respectively $\bar{\rho}_j^{n+1} \geqslant \bar{\rho}_{j+1}^{n+1}$), then $\rho_j^{n+1} \leqslant \rho_{j+1}^{n+1}$ (respectively $\rho_j^{n+1} \geqslant \rho_{j+1}^{n+1}$).

Remarks. 1. Boris and Book introduced the explicit antidiffusion flux $\bar{f}_{j+1/2}^{n+1}$ such that

$$\bar{f}_{j+1/2}^{n+1} = s \cdot \mathrm{Max}\left[0, \mathrm{Min}\left(s \cdot \Delta_{j-1/2}^{n+1}, \frac{1}{8}|\Delta_{j+1/2}^{n+1}|, s \cdot \Delta_{j+3/2}^{n+1}\right)\right]$$

in which s denotes the sign of $\Delta_{j+1/2}^{n+1}$. But this does not satisfy the condition (e).

2. The restriction that $K_0 \leqslant 1/8$ is very important numerically, but we need only that $K_0 \leqslant 1$ in the following sections:

LEMMA 1. *We have*

$$\rho_j^{n+1} = \bar{\rho}_j^{n+1} \quad \text{if } \bar{\rho}_j^{n+1} \geqslant \mathrm{Max}(\bar{\rho}_{j-1}^{n+1}, \bar{\rho}_{j+1}^{n+1}) \text{ or if } \bar{\rho}_j^{n+1} \leqslant \mathrm{Min}(\bar{\rho}_{j-1}^{n+1}, \bar{\rho}_{j+1}^{n+1}), \tag{8}$$

$$|\rho_j^{n+1} - \bar{\rho}_j^{n+1}| \leqslant \mathrm{Min}(|\Delta_{j-1/2}^{n+1}|, |\Delta_{j+1/2}^{n+1}|), \tag{9}$$

and

$$\mathrm{Min}(\bar{\rho}_{j-1}^{n+1}, \bar{\rho}_j^{n+1}, \bar{\rho}_{j+1}^{n+1}) \leqslant \rho_j^{n+1} \leqslant \mathrm{Max}(\bar{\rho}_{j-1}^{n+1}, \bar{\rho}_j^{n+1}, \bar{\rho}_{j+1}^{n+1}). \tag{10}$$

(*The estimation* (10) *means that the operation* (7) *generates no new maxima or minima.*)

Proof. Omitted.

Examples of the Antidiffusion Flux.

Example 1.

$$f_{j+1/2}^{n+1} = s \cdot \mathrm{Max}\left[0, \mathrm{Min}\left(\frac{5}{8} s \cdot \Delta_{j-1/2}^{n+1}, \frac{1}{8}|\Delta_{j+1/2}^{n+1}|, \frac{5}{8} s \cdot \Delta_{j+3/2}^{n+1}\right)\right], \tag{11}$$

where s denotes the sign of $\Delta_{j+1/2}^{n+1}$.

Example 2.

$$f_{j+1/2}^{n+1} = \bar{f}_{j+1/2}^{n+1} \cdot \mathrm{Min}(1, \alpha_{j-1/2}^{n+1}, \alpha_{j+3/2}^{n+1}), \tag{12}$$

where $\bar{f}_{j+1/2}^{n+1}$ is the explicit antidiffusion flux introduced by Boris and Book, and

$$\alpha_{j+1/2}^{n+1} = \begin{cases} 1 & \text{if } \bar{f}_{j-1/2}^{n+1} + \bar{f}_{j+3/2}^{n+1} = 0, \\ |\Delta_{j+1/2}^{n+1} + 2\bar{f}_{j+1/2}^{n+1}|/|\bar{f}_{j-1/2}^{n+1} + \bar{f}_{j+3/2}^{n+1}| & \text{otherwise.} \end{cases}$$

3. **Stability**. In what follows, we shall always assume that $v(\rho)$ is a continuously differentiable real-valued function of the single real variable ρ and that $\rho_0(x)$ is a bounded function. We denote the infimum (respectively supremum) of $\rho_0(x)$ by s (respectively S). Let $V_{s,S}$ be the absolute maximum of $dv(\rho)/d\rho$ in $s \leqslant \rho \leqslant S$.

LEMMA 2. *Suppose that $s \leqslant \xi \leqslant S$ and $s \leqslant \eta \leqslant S$. Then we have the estimates*

$$0 \leqslant \frac{\partial F}{\partial \xi}(\xi, \eta) \leqslant \frac{1}{2}, \qquad -\frac{1}{2} \leqslant \frac{\partial F}{\partial \eta}(\xi, \eta) \leqslant 0 \tag{13}$$

provided that

$$\lambda[V_{s,S}(S - s + 4\,\mathrm{Max}(|s|, |S|)) + 2|v((s + S)/2)|] \leqslant 1. \tag{14}$$

(*In the case of $v(\rho) \equiv v =$ constant, this inequality is written as $\lambda \leqslant 1/2v$.*)

Proof. For simplicity, we put $p = V_{s,S}(S - s)/2$, $q = V_{s,S}\,\mathrm{Max}(|s|, |S|)$, and $r = |v((s + S)/2)|$. Then the condition (14) is equivalent to

$$2\lambda[p + 2q + r] \leqslant 1. \tag{14$'$}$$

We have

$$\frac{\partial E}{\partial \xi} = (1 + E)\frac{\lambda \dot{v}(\xi)}{1 + \lambda v(\eta) - \lambda v(\xi)}, \qquad \frac{\partial E}{\partial \eta} = (1 - E)\frac{\lambda \dot{v}(\eta)}{1 + \lambda v(\eta) - \lambda v(\xi)}.$$

Therefore, the partial derivatives $\partial F/\partial \xi$ and $\partial F/\partial \eta$ are written in the form

$$\frac{\partial F}{\partial \xi} = \frac{1 + E}{8}\left[(1 + E) + \frac{2\lambda \dot{v}(\xi)}{1 + \lambda v(\eta) - \lambda v(\xi)}\{(1 + E)\xi + (1 - E)\eta\}\right], \tag{15}$$

$$\frac{\partial F}{\partial \eta} = -\frac{1 - E}{8}\left[(1 - E) - \frac{2\lambda \dot{v}(\eta)}{1 + \lambda v(\eta) - \lambda v(\xi)}\{(1 + E)\xi + (1 - E)\eta\}\right]. \tag{16}$$

Put $X = ½ - \lambda v(\xi)$, $Y = ½ + \lambda v(\eta)$, $A = \lambda \dot{v}(\xi)\xi$, $B = \lambda \dot{v}(\xi)\eta$, $C = \lambda \dot{v}(\eta)\xi$, and $D = \lambda \dot{v}(\eta)\eta$, respectively. By virtue of the fact that $1 + E = 2Y/(X + Y)$ and $1 - E = 2X/(X + Y)$, (15) and (16) are rewritten as

$$\frac{\partial F}{\partial \xi} = \frac{Y}{2(X + Y)^3}[(Y + 2B)X + (Y + 2A)Y], \tag{17}$$

$$\frac{\partial F}{\partial \eta} = -\frac{X}{2(X+Y)^3}\,[(X-2D)X + (X-2C)Y]. \tag{18}$$

Hence, the family of inequalities

$$\begin{cases} 0 \leqslant \tfrac{1}{2} - \lambda p - \lambda r \leqslant X \leqslant \tfrac{1}{2} + \lambda p + \lambda r, \\ 0 \leqslant \tfrac{1}{2} - \lambda p - \lambda r \leqslant Y \leqslant \tfrac{1}{2} + \lambda p + \lambda r, \\ 0 < 1 - 2\lambda p \leqslant X + Y, \\ |A| \leqslant \lambda q, \quad |B| \leqslant \lambda q, \quad |C| \leqslant \lambda q, \quad |D| \leqslant \lambda q, \end{cases}$$

implies that $\partial F/\partial \xi \geqslant 0$ and $\partial F/\partial \eta \leqslant 0$ under the condition of (14′). If we regard X, Y, A, B, C, and D as independent variables, the right-hand side of (17) (respectively (18)) is monotone nondecreasing in A and B (respectively C and D). Therefore, we have, under the restriction of (14′),

$$\begin{cases} \dfrac{\partial F}{\partial \xi} \leqslant \dfrac{1}{2} \cdot \dfrac{Y(Y+2\lambda q)}{(X+Y)^2} \leqslant \dfrac{1}{2}\left(\dfrac{1}{2} + \lambda p + \lambda r\right)\left(\dfrac{1}{2} + \lambda p + 2\lambda q + \lambda r\right) \leqslant \dfrac{1}{2}, \\ \dfrac{\partial F}{\partial \eta} \geqslant -\dfrac{1}{2} \cdot \dfrac{X(X+2\lambda q)}{(X+Y)^2} \geqslant -\dfrac{1}{2}\left(\dfrac{1}{2} + \lambda p + \lambda r\right)\left(\dfrac{1}{2} + \lambda p + 2\lambda q + \lambda r\right) \geqslant -\dfrac{1}{2}, \end{cases}$$

as $Y(Y+2\lambda q)/(X+Y)^2$ (respectively $X(X+2\lambda q)/(X+Y)^2$) is monotone decreasing in X (respectively Y) and monotone nondecreasing in Y (respectively X). This completes the proof. Q.E.D.

On account of the above lemma, the SHASTA is a positive finite difference scheme if (14) is observed. Moreover, the antidiffusion operation generates no new maxima or minima, so we obtain the following theorem.

THEOREM 1. *Under the condition of* (14) *the* SHASTA FCT *algorithm is* L^∞-*stable, and it holds that*

$$\inf \rho_0(x) \leqslant \rho_j^k \leqslant \sup \rho_0(x) \tag{19}$$

for any j and nonnegative k.

4. Convergence of a Subsequence of $\{\rho_h(x, t)\}$. The purpose of this section is to show that there exists a subsequence of $\{\rho_h(x, t)\}$ tending to one of the generalized solutions to (CP) in the L^1_{loc}-sense under some assumptions. Hereafter, we shall always assume that ρ_0 is a measurable function having locally bounded variation. We fix λ to satisfy (14). Let us use the notation Vari$(X; \nu)$ to denote the total variation in $[-X, X]$ of $\nu = \nu(x)$ which is defined in $\mathbf{R}$ having locally bounded variation.

Now, the following fact is well known.

LEMMA 3 (OLEINIK [15]). *Let $\{h_n\}$ be a sequence such that*

$$h_n > 0 \quad \textit{for } n = 1, 2, \ldots, \quad \lim_{n\to\infty} h_n = 0.$$

Suppose that the sequence of real-valued functions $\{\mu_{h_n}(x, t)\}_{n=1,2,\ldots}$ *defined in* $\mathbf{R} \times \mathbf{R}^+$ *satisfies the following conditions*:

(i) *Each of* $\{\mu_{h_n}(x, t)\}_{n=1,2,\ldots}$ *is a bounded measurable function, and the absolute values of* $\mu_{h_n}(x, t)$ *are uniformly bounded in* h_n.

(ii) *Each of* $\{\mu_{h_n}(x, t)\}_{n=1,2,\ldots}$ *is of locally bounded variation as a function of x, and for any fixed* $T > 0$ *and* $X > 0$ *the total variations* $\mathrm{Vari}(X; \mu_{h_n}(\cdot, t))$ *are uniformly bounded in* h_n *and* $0 \leqslant t \leqslant T$.

(iii) *For arbitrary fixed* $T > 0$ *and* $X > 0$, *there exists a constant C independent of* h_n *so that the estimation*

$$\int_{-X}^{X} |\mu_{h_n}(x, t) - \mu_{h_n}(x, t')|\, dx \leqslant C \cdot (|t - t'| + h_n)$$

holds for any $0 \leqslant t \leqslant T$ *and* $0 \leqslant t' \leqslant T$.

Then, there exists a pair of a subsequence $\{\mu_{h'_m}(x, t)\}_{m=1,2,\ldots}$ *of* $\{\mu_{h_n}(x, t)\}_{n=1,2,\ldots}$ *and a bounded measurable function* $\mu(x, t)$, *which satisfies the following properties*:

(iv) $|\mu(x, t)| \leqslant \sup_{m,x',t'} |\mu_{h'_m}(x', t')|$.

(v) $\lim_{m\to\infty} \int_{-X}^{X} |\mu_{h'_m}(x, t) - \mu(x, t)|\, dx = 0$ *if* $0 \leqslant t \leqslant T$ *for any fixed* $T > 0$ *and* $X > 0$.

(vi) $\lim_{m\to\infty} \int_0^T \int_{-X}^{X} |\mu_{h'_m}(x, t) - \mu(x, t)|\, dx\, dt = 0$ *for any fixed* $T > 0$ *and* $X > 0$.

Proof. Omitted.

In the present case we see, as a result of Theorem 1, that the sequence $\{\rho_h(x, t)\}$ satisfies the condition (i) in Lemma 3. Let us establish the conditions (ii) and (iii) to $\{\rho_h(x, t)\}$.

LEMMA 4. *For fixed* $T > 0$ *and* $X > 0$, *we have the estimate*

$$\sum_{j=-J}^{J-1} |\rho_{j+1}^k - \rho_j^k| \leqslant \mathrm{Vari}(X + 2T/\lambda; \rho_0), \tag{20}$$

where $Jh \leqslant X$ *and* $0 \leqslant k\tau \leqslant T$.

Proof. From (d) and (e) it follows that

$$\sum_{j=-J}^{J-1} |\rho_{j+1}^k - \rho_j^k| \leqslant |\rho_{-J}^k - \bar\rho_{-J}^k| + \sum_{j=-J}^{J-1} |\bar\rho_{j+1}^k - \bar\rho_j^k| + |\rho_J^k - \bar\rho_J^k|.$$

Hence, we obtain $\Sigma_{j=-J}^{J-1} |\rho_{j+1}^k - \rho_j^k| \leqslant \Sigma_{j=-J-1}^{J} |\bar\rho_{j+1}^k - \bar\rho_j^k|$ by (9). We put $w_j^k = \rho_{j+1}^k - \rho_j^k$ and $\bar w_j^k = \bar\rho_{j+1}^k - \bar\rho_j^k$. From (6) we obtain, by applying the mean value theorem,

$$\begin{aligned} \bar w_j^k = &\left[1 - \frac{\partial F}{\partial \xi}(\phi_{j+1/2}^{k-1}, \rho_{j+1}^{k-1}) + \frac{\partial F}{\partial \eta}(\rho_j^{k-1}, \psi_{j+1/2}^{k-1})\right] \cdot w_j^{k-1} \\ &+ \left[\frac{\partial F}{\partial \xi}(\phi_{j-1/2}^{k-1}, \rho_j^{k-1})\right] \cdot w_{j-1}^{k-1} + \left[-\frac{\partial F}{\partial \eta}(\rho_{j+1}^{k-1}, \psi_{j+3/2}^{k-1})\right] \cdot w_{j+1}^{k-1}, \end{aligned} \tag{21}$$

where $\phi_{j+1/2}^{k-1}$ and $\psi_{j+1/2}^{k-1}$ are two intermediate values between ρ_j^{k-1} and ρ_{j+1}^{k-1}. The three coefficients of the w_j^{k-1} are nonnegative due to Lemma 2. Therefore, it holds that

$$\sum_{j=-J-1}^{J} |\bar{w}_j^k| \leqslant \sum_{j=-J-2}^{J+1} |w_j^{k-1}|$$

which results in

$$\sum_{j=-J}^{J-1} |\rho_{j+1}^k - \rho_j^k| \leqslant \sum_{j=-J-2}^{J+1} |\rho_{j+1}^{k-1} - \rho_j^{k-1}|.$$

By continuing in this way k times, we obtain the desired estimate

$$\sum_{j=-J}^{J-1} |\rho_{j+1}^k - \rho_j^k| \leqslant \sum_{j=-J-2k}^{J+2k-1} |\rho_{j+1}^0 - \rho_j^0| \leqslant \mathrm{Vari}(X + 2T/\lambda; \rho_0). \quad \text{Q.E.D.}$$

LEMMA 5. *Fix $T > 0$ and $X > 0$ arbitrary. Then, if $0 \leqslant l\tau \leqslant k\tau \leqslant T$ and $Jh \leqslant X$, we have*

$$\sum_{j=-J}^{J-1} |\rho_j^k - \rho_j^l| h \leqslant C \cdot (k - l)h, \tag{22}$$

where the constant C is defined by $C = (1 + 2K_0) \cdot \mathrm{Vari}(X + 2T/\lambda; \rho_0)$.

Proof. By making use of the triangle inequality we have

$$\sum_{j=-J}^{J-1} |\rho_j^{n+1} - \rho_j^n| \leqslant \sum_{j=-J}^{J-1} |\rho_j^{n+1} - \bar{\rho}_j^{n+1}| + \sum_{j=-J}^{J-1} |\bar{\rho}_j^{n+1} - \rho_j^n|.$$

The first term of the right-hand side is estimated as

$$\sum_{j=-J}^{J-1} |\rho_j^{n+1} - \bar{\rho}_j^{n+1}| \leqslant \sum_{j=-J}^{J-1} |f_{j+1/2}^{n+1} - f_{j-1/2}^{n+1}| \leqslant 2 \sum_{j=-J-1}^{J-1} |f_{j+1/2}^{n+1}|$$

$$\leqslant 2K_0 \sum_{j=-J-1}^{J-1} |\bar{\rho}_{j+1}^{n+1} - \bar{\rho}_j^{n+1}| \leqslant 2K_0 \sum_{j=-J-2}^{J} |\rho_{j+1}^n - \rho_j^n|$$

in the same way as in the proof of Lemma 4. On the other hand, from (6) we obtain, by applying the mean value theorem and Lemma 2,

$$|\bar{\rho}_j^{n+1} - \rho_j^n| \leqslant \frac{1}{2} |\rho_{j+1}^n - \rho_j^n| + \frac{1}{2} |\rho_j^n - \rho_{j-1}^n|,$$

and this results in

$$\sum_{j=-J}^{J-1} |\bar{\rho}_j^{n+1} - \rho_j^n| \leqslant \sum_{j=-J-1}^{J-1} |\rho_{j+1}^n - \rho_j^n|.$$

Therefore, we have

$$\sum_{j=-J}^{J-1} |\rho_j^{n+1} - \rho_j^n| h \leqslant (1 + 2K_0) \sum_{j=-J-2}^{J} |\rho_{j+1}^n - \rho_j^n| h$$

$$\leqslant (1 + 2K_0)\, \mathrm{Vari}(X + 2T/\lambda; \rho_0) h$$

by Lemma 4 for $n = l, l + 1, \ldots, k - 1$. This completes the proof. Q.E.D.

Lemma 4 (respectively Lemma 5) assures that $\{\rho_h(x, t)\}$ satisfies the condition (ii) (respectively (iii)) in Lemma 3. Hence, by virtue of Lemma 3, there exists a pair of a subsequence $\{\rho_{h'}(x, t)\}$ and the limit function $\rho^*(x, t)$ satisfying the properties (iv), (v), and (vi). Moreover, it is concluded that this limit function is one of the generalized solutions to (CP), that is, it holds that

$$\iint_{t \geqslant 0} [\rho^* \zeta_t + v(\rho^*)\rho^* \zeta_x]\, dx\, dt + \int_{t=0} \rho_0 \zeta\, dx = 0$$

for all smooth functions $\zeta = \zeta(x, t)$ having compact support. (See Lax-Wendroff's Theorem [12]. This theorem is applicable to the present case since the SHASTA FCT algorithm is in conservation form and consistent with (CP).) Thus, we obtain the main theorem.

THEOREM 2. *One can choose from* $\{\rho_h(x, t)\}$ *a subsequence* $\{\rho_{h'}(x, t)\}$ *which converges to a generalized solution to* (CP) *in* $L^1_{\mathrm{loc}}(\mathbf{R} \times \mathbf{R}^+)$. *Hence, the* SHASTA FCT *solution is convergent provided that the generalized solution to* (CP) *exists uniquely.*

Proof. The first part has been verified already. The second part is shown by means of reduction to absurdity. Q.E.D.

It is shown that the solution satisfying the entropy condition in Kružkov's sense exists uniquely. (See Kružkov [10].) Since the physically relevant solution must satisfy the entropy condition, it is important not only theoretically but also physically and practically whether or not the limit function of the SHASTA FCT algorithm satisfies the entropy condition. In some numerical experiments, the SHASTA FCT seems to converge to the solution to (CP) satisfying the entropy condition, but we cannot yet prove whether or not it is true. However, if the calculation would be done without the FCT operation, the limit of the numerical solutions with the SHASTA only is the physically relevant solution. This is easily shown by a similar argument as in the case of the SHASTA FCT on the ground that the SHASTA is a positive scheme. (See [9].) By observing this fact, we may introduce the following technical modification into the FCT operation.

Let $\gamma(h)$ be a function of h satisfying the condition that

$$0 < \gamma(h) \leqslant 1 \quad \text{and} \quad \lim_{h \to \infty} \gamma(h) = 0,$$

and let $\tilde{f}^{n+1}_{j+1/2}$ be

$$\tilde{f}^{n+1}_{j+1/2} = \gamma(h) f^{n+1}_{j+1/2}.$$

Then, the modified antidiffusion is

$$\rho^{n+1}_j = \bar{\rho}^{n+1}_j - \tilde{f}^{n+1}_{j+1/2} + \tilde{f}^{n+1}_{j-1/2}. \tag{7'}$$

Then, we have

THEOREM 3. *The solution with the* SHASTA FCT *algorithm with* $\tilde{f}^{n+1}_{j+1/2}$ *in* *place of* $f_{j+1/2}^{n+1}$ *is convergent and the limit function is the physically relevant solution to* (CP). □

Remark. The convergence rate of $\gamma(h)$ as $h \downarrow 0$ may be arbitrarily slow.

Proof of Theorem 3. Let $\{\hat{\rho}_h(x, t)\}$ be the sequence of finite difference solutions with the SHASTA only. By Lemma 6 which follows, it holds that, for any fixed $T > 0$ and $X > 0$,

$$\sum_{j=-J}^{J-1} |\hat{\rho}_j^{n+1} - \rho_j^{n+1}|h \leqslant \sum_{j=-J}^{J-1} |\hat{\rho}_j^{n+1} - \bar{\rho}_j^{n+1}|h + \sum_{j=-J}^{J-1} |\bar{\rho}_j^{n+1} - \rho_j^{n+1}|h$$

$$\leqslant \sum_{j=-J-1}^{J} |\hat{\rho}_j^n - \rho_j^n|h + 2\gamma(h)K_0 \operatorname{Vari}(X + 2T/\lambda; \rho_0)h$$

if $Jh \leqslant X$ and $(n + 1)\tau \leqslant T$. The above estimation implies that ρ_h tends to $\hat{\rho}_h$ as h tends to 0, that is,

$$\lim_{h \downarrow 0} \int_{-X}^{X} |\rho_h(x, t) - \hat{\rho}_h(x, t)|\, dx = 0 \quad \text{if } t \leqslant T,$$

$$\lim_{h \downarrow 0} \int_0^T \int_{-X}^{X} |\rho_h(x, t) - \hat{\rho}_h(x, t)|\, dx\, dt = 0$$

for arbitrary fixed $T > 0$ and $X > 0$. And so, we obtain the proof. Q.E.D.

LEMMA 6 (B. KEYFITZ [9]). *If the finite difference scheme*

$$u_j^{n+1} = H(u_{j-k}^n, \ldots, u_{j+k}^n)$$

is positive and in conservation form, then it holds that

$$\sum_{j=-J}^{J-1} |u_j^{n+1} - w_j^{n+1}| \leqslant \sum_{j=-J-k}^{J+k-1} |u_j^n - w_j^n|$$

for any $\{u_j^n\}$ *and any* $\{w_j^n\}$.

Proof. Omitted.

5. **Numerical Examples.** We compare numerically the algorithm defined by (6), (7), and (11), which will be denoted by SHASTA-FCT$_1$, with original SHASTA FCT algorithm. Test problems are the linear equation $\partial\rho/\partial t + \partial\rho/\partial x = 0$ and the Burgers equation $\partial\rho/\partial t + \partial(\rho^2/2)/\partial x = 0$. Figures 2 to 5 show the profiles of the computed solutions. We remark that

(1) There is less distinction between the numerical solution by SHASTA-FCT$_1$ and the solution by the original SHASTA FCT algorithm.

(2) The numerical solutions by both FCT algorithms are better than the ones by Lax-Wendroff's scheme.

(3) If the condition of (14) is not satisfied, the SHASTA FCT algorithms yield in some cases numerical solutions which are far off the exact solutions. (See Figure 5(b).)

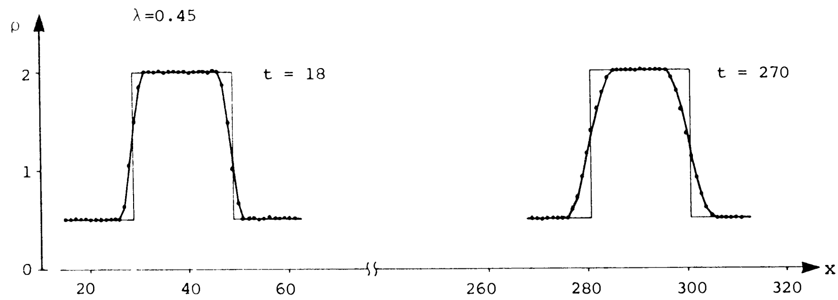


FIGURE 2

Profiles of numerical solutions for the linear problem:

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial x}=0,\qquad \rho_0(x)=\begin{cases}2.0 & \text{for } |x-20.5|<10,\\ 0.5 & \text{otherwise.}\end{cases}$$

The fine lines are the exact solution and the thick lines consisting of piecewise linear segments are the interpolated solution by SHASTA-FCT$_1$. The dots are the values computed by the original SHASTA FCT algorithm.

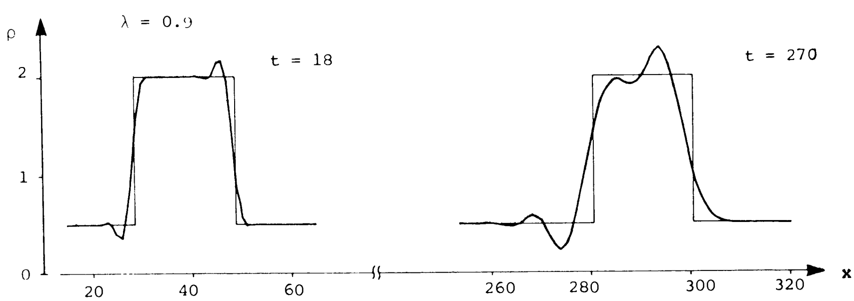


FIGURE 3

Profiles of numerical solutions by Lax-Wendroff's scheme for the same problem as in Figure 2. The fine lines are the exact solution and the thick lines are the solution computed by Lax-Wendroff's scheme.

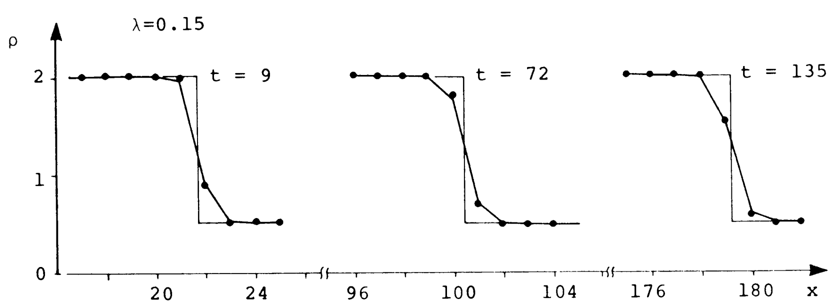


FIGURE 4

Profiles of numerical solutions for the nonlinear problem:

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}\left(\frac{\rho^2}{2}\right)=0,\qquad \rho_0(x)=\begin{cases}2.0 & \text{for } x<10.5,\\ 0.5 & \text{for } x>10.5.\end{cases}$$

See Figure 2 for explanation of these curves.

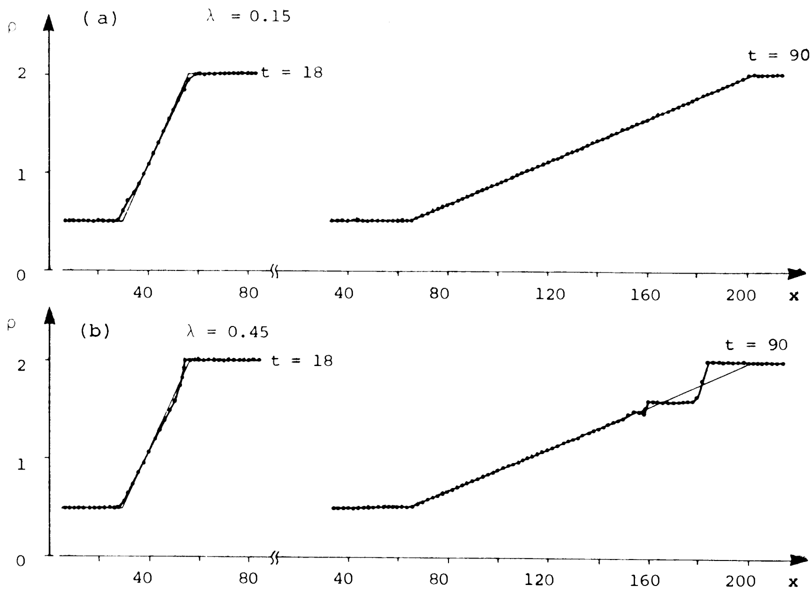


FIGURE 5

Profiles of numerical solutions for the nonlinear problem:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\rho^2}{2}\right) = 0, \qquad \rho_0(x) = \begin{cases} 0.5 & \text{for } x < 20.5, \\ 2.0 & \text{for } x > 20.5, \end{cases}$$

under the two different values of the mesh ratio $\lambda = \tau/h$. See Figure 2 for explanation of these curves. The condition of (14) is written as $\lambda \leqslant 1/6$ in this case. Figure 5(b) shows that the SHASTA FCT algorithm yields the solution far off the exact one if (14) is violated.

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